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Several remarks on dimensions modulo *ANR*-compacta

V.V. Fedorchuk [1]

Moscow State University (M.V. Lomonosov), Department of General Topology and Geometry, Moscow, Russia

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ABSTRACT

We investigate a dimension function $\mathcal{L}$-dim ($\mathcal{L}$ is a class of *ANR*-compacta). Main results are as follows.
Let L be an *ANR*-compactum.
(1) If $L * L$ is not contractible, then for every $n \geqslant 0$ there is a cube I^m with $L\text{-}\dim I^m = n$.
(2) If L is simply connected and $f : X \to Y$ is an acyclic mapping from a finite-dimensional compact Hausdorff space X onto a finite-dimensional space Y, then $L\text{-}\dim Y \leqslant L\text{-}\dim X$.
(3) If L is simply connected and $L * L$ is not contractible, then for every $n \geqslant 2$ there exists a compact Hausdorff space Z_n^L such that $L\text{-}\dim Z_n^L = n$, and for an arbitrary closed set $F \subset Z_n^L$ either $L\text{-}\dim F \leqslant 0$ or $L\text{-}\dim F = n$.

0. Introduction

In [10] new dimension functions $\mathcal{K}$-dim and $\mathcal{L}$-dim were introduced where $\mathcal{K}$ is a non-empty class of finite simplicial complexes and $\mathcal{L}$ is a non-empty class of *ANR*-compacta. These definitions are based on the theorem on partitions (see Definition 1.5) and on the theorem on mappings to spheres (see Definition 1.7). If $\mathcal{K} = \mathcal{L} = \{S^0\}$, then

$$\mathcal{K}\text{-}\dim X = \mathcal{L}\text{-}\dim X = \dim X \tag{0.1}$$

for an arbitrary normal space X. For arbitrary classes $\mathcal{K}$ and $\mathcal{L}$ we have

$$\mathcal{K}\text{-}\dim X \leqslant \dim X, \qquad \mathcal{L}\text{-}\dim X \leqslant \dim X. \tag{0.2}$$

If $\mathcal{L}$ is a class of polyhedra, τ is an arbitrary triangulation of this class, and $\mathcal{L}_\tau$ is the corresponding class of simplicial complexes, then

$$\mathcal{L}_\tau\text{-}\dim X = \mathcal{L}\text{-}\dim X. \tag{0.3}$$

In the present paper we continue an investigation of these dimension functions. In view of (0.3) it suffices to consider only the function $\mathcal{L}$-dim. In Section 2 we study dimensional scales of *ANR*-compacta. Theorem 2.5 states that if $L * L$ is not contractible, then for every $n \geqslant 0$ there is a cube I^m such that $L\text{-}\dim \dot{I}^m = n$.

E-mail address: vvfedorchuk@gmail.com.

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The main result of Section 4 is

4.7. Theorem. *Let L be an ANR-compactum and let $f : X \to Y$ be an acyclic mapping from a finite-dimensional compact Hausdorff space X onto a finite-dimensional space Y. If $L\text{-}\dim Y \neq 1$, then $L\text{-}\dim X \geqslant L\text{-}\dim Y$. If $L\text{-}\dim Y = 1$, then $L\text{-}\dim X \geqslant 1$ for a simply connected L.*

In Section 5 we construct a compact Hausdorff spaces Z_n^L of dimension $L\text{-}\dim Z_n^L = n$ with no intermediate L-dimensions where $n \geqslant 2$ and L is a simply connected *ANR*-compactum with non-contractible $L * L$. For Lebesgue dimension dim such spaces were constructed in [4]. To construct spaces Z_n^L we apply fully closed resolutions. In Section 3 we recall necessary facts concerning this area. All mappings are assumed to be continuous.

1. Preliminaries

A metrizable compact Hausdorff space is called a *compactum*.

1.1. By a *space* we mean a normal $(+T_1)$ topological space. For a space X by $\exp X$ we denote the set of all closed subsets of X (including $\emptyset$). By $\mathrm{Fin}_s(\exp X)$ we denote the set of all finite sequences $\Phi = (F_1, \dots, F_m)$, $F_j \in \exp X$.

In what follows *complexes* stand for finite complete simplicial complexes. Sometimes we identify complexes with underlying polyhedra. Hence every complex is an *ANE*-space (for normal spaces). In what follows *polyhedra* stand for compact polyhedra.

For a complex K by $v(K)$ we denote the set of all its vertices. Let u be a finite family of sets and let $u_0 = \{U \in u\colon\ U \neq \emptyset\}$. The *nerve* of the family u is a complex $N(u)$ such that $v(N(u)) = \{a_U\colon\ U \in u_0\}$ and a set $\Delta \subset v\bigl(N(u)\bigr)$ is a simplex of $N(u)$ if and only if $\bigcap\{U\colon\ a_U \in \Delta\} \neq \emptyset$.

1.2. Definition. ([9]) Let X be a space, K be a complex, and $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$. A sequence $u = (U_1, \dots, U_k)$, $k \geqslant m$, of open subsets of X is called a *K-neighborhood* of Φ if $F_j \subset U_j$ and there is an embedding $N(u) \subset K$. One can number vertices $a_j \in v(K)$ so that the embedding $N(u) \subset K$ is defined by the correspondence $U_j \to a_j$.

1.3. Definition. ([9]) A set $P \subset X$ is said to be a *K-partition* of $\Phi \in \mathrm{Fin}_s(\exp X)$ (notation: $P \in \mathrm{Part}(\Phi, K)$) if $P = X \setminus \bigcup u$ where u is a K-neighborhood of Φ.

Put $\mathrm{Exp}_K(X) = \{\Phi \in \mathrm{Fin}_s(\exp X)\colon\ N(\Phi) \subset K\}$.

1.4. Definition. A sequence $(K_1, \dots, K_r)$ of complexes is called *inessential* in X if for every sequence $(\Phi_1, \dots, \Phi_r)$ such that $\Phi_i \in \mathrm{Exp}_{K_i}(X)$ there exist K_i-partitions P_i of Φ_i such that $P_1 \cap \dots \cap P_r = \emptyset$.

Partitions P_i correspond to K_i-neighborhoods $u_i = (U_1^i, \dots, U_{K_i}^i)$ of $\Phi_i = (F_1^i, \dots, F_{m_i}^i)$ from Definition 1.2. It was shown in [10] that we can require that embeddings $N(u_i) \subset K_i$ extend embeddings $N(\Phi_i) \subset K_i$ from the definition of the set $\mathrm{Exp}_{K_i}(X)$.

1.5. Definition. ([10]) Let $\mathcal{K}$ be a class of complexes. To every space X one assigns the dimension $\mathcal{K}\text{-}\dim X$ which is an integer $\geqslant -1$ or ∞. The dimension function $\mathcal{K}$-dim is defined in the following way:

(1) $\mathcal{K}\text{-}\dim X = -1 \iff X = \emptyset$;
(2) $\mathcal{K}\text{-}\dim X \leqslant n$, where $n = 0, 1, \dots$, if every sequence $(K_1, \dots, K_{n+1})$, $K_i \in \mathcal{K}$, $i = 1, \dots, n+1$, is inessential in X; and
(3) $\mathcal{K}\text{-}\dim X = \infty$, if $\mathcal{K}\text{-}\dim X > n$ for all $n = -1, 0, 1, \dots$.

If the class $\mathcal{K}$ contains only one complex K we write $\mathcal{K} = K$ and $\mathcal{K}\text{-}\dim X = K\text{-}\dim X$.

Hemmingsen's theorem on partitions yields

1.6. Theorem. $\{0, 1\}\text{-}\dim X = \dim X$.

In what follows $\mathcal{L}$ stands for a non-empty class of metrizable *ANR*-compacta L. By $X_1 * \dots * X_n$ we denote the join of spaces $X_1, \dots, X_n$.

1.7. Definition. To every space X one assigns the dimension $\mathcal{L}\text{-}\dim X$ which is an integer $\geqslant -1$ or ∞. The dimension function $\mathcal{L}$-dim is defined in the following way:

(1) $\mathcal{L}\text{-}\dim X = -1 \iff X = \emptyset$;
(2) $\mathcal{L}\text{-}\dim X \leqslant n$, where $n = 0, 1, \dots$, if $L_1 * \dots * L_{n+1} \in AE(X)$ for any $L_1, \dots, L_{n+1} \in \mathcal{L}$; and
(3) $\mathcal{L}\text{-}\dim X = \infty$, if $\mathcal{L}\text{-}\dim X > n$ for all $n \geqslant -1$.

If the class $\mathcal{L}$ contains only one compactum L we write $\mathcal{L} = L$ and $\mathcal{L}\text{-}\dim X = L\text{-}\dim X$.

Since $S^n = \overset{n+1}{*} S^0$, from a characterization of the Lebesgue dimension by means of mappings to spheres we get

1.8. Theorem. *For every space X, S^0-$\dim X = \dim X$.*

1.9. Theorem. *([10]) Let X be a separable metrizable space such that $\mathcal{L}$-$\dim X \leqslant n$. Then X can be represented as the union of $n+1$ subspaces $X_1, \dots, X_{n+1}$ so that $\mathcal{L}$-$\dim X_i \leqslant 0$ for $i = 1, \dots, n+1$.*

Let $\mathcal{L}$ be a non-empty class of polyhedra. For each $L \in \mathcal{L}$ we fix a triangulation $t = t(L)$ of L. The pair (L, t) is a simplicial complex which is denoted by L_t. The family $\tau = \{t(L)\colon\ L \in \mathcal{L}\}$ is said to be a *triangulation* of the class $\mathcal{L}$. Let $\mathcal{L}_\tau = \{L_t\colon\ t \in \tau\}$.

1.10. Theorem. *([10]) Let $\mathcal{L}$ be a class of polyhedra and let τ be some its triangulation. Then $\mathcal{L}$-$\dim X = \mathcal{L}_\tau$-$\dim X$ for every space X.*

Since *ANR*-compacta are *ANE*'s for normal spaces, we have

1.11. Proposition. *Let F be a closed subspace of a space X such that $\mathcal{L}$-$\dim F \leqslant n$ and $\mathcal{L}$-$\dim E \leqslant n$ for any closed subset $E \subset X$ with $E \cap F = \emptyset$. Then $\mathcal{L}$-$\dim X \leqslant n$.*

We say that a class $\mathcal{L}_1$ *is* (*homotopically*) *dominated by* a class $\mathcal{L}_2$ (notation: $\mathcal{L}_1 \leqslant_h \mathcal{L}_2$) if every $L_1 \in \mathcal{L}_1$ is dominated by some $L_2 \in \mathcal{L}_2$. A class $\mathcal{L}_1$ is homotopically *equivalent* to a class $\mathcal{L}_2$ (notation: $\mathcal{L}_1 \simeq \mathcal{L}_2$) if both $\mathcal{L}_1 \leqslant_h \mathcal{L}_2$ and $\mathcal{L}_2 \leqslant_h \mathcal{L}_1$ hold.

1.12. Proposition. *If $\mathcal{L}_1 \leqslant_h \mathcal{L}_2$, then $\mathcal{L}_1$-$\dim X \leqslant \mathcal{L}_2$-$\dim X$ for every X.*

As a corollary we get

1.13. Proposition. *If $\mathcal{L}_1 \simeq \mathcal{L}_2$, then $\mathcal{L}_1$-$\dim X = \mathcal{L}_2$-$\dim X$ for every X.*

In view of Theorem 1.10, Proposition 1.13, and West's theorem on homotopy types of *ANR*-compacta one can consider only dimension functions $\mathcal{L}$-dim with $\mathcal{L}$ consisting of polyhedra and apply results of extension theory.

1.14. Theorem. *([10]) If X is the limit space of an inverse system $\{X_\alpha, \pi^\alpha_\beta, A\}$ of compact Hausdorff spaces X_α such that $\mathcal{L}$-$\dim X_\alpha \leqslant n$, then $\mathcal{L}$-$\dim X \leqslant n$.*

Recall that an inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ is said to be *continuous* if, for each chain B in A with $\sup B = \beta \in A$, the diagonal product $\triangle\{\pi^\beta_\alpha\colon\ \alpha \in B\}$ maps the space X_β homemorphically onto the space $\lim(S|B)$.

An inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ is called a *σ-spectrum* [14] if

(1) all X_α are compacta;
(2) the set A is *ω-complete*, that is for every countable chain $B \subset A$ there is $\sup B \in A$; and
(3) the system S is *ω-continuous*, i.e. a continuity condition is fulfilled for countable chains $B \subset A$.

1.15. Theorem. *([1,10]) Let X be a compact Hausdorff space and let L be an ANR-compactum such that L-$\dim X \leqslant n$. Then X is the limit space of a σ-spectrum $\{X_\alpha, \pi^\alpha_\beta, A\}$ such that L-$\dim X_\alpha \leqslant n$ for every $\alpha \in A$.*

1.16. Theorem. *([10]) $\mathcal{L}$-$\dim X = \mathcal{L}$-$\dim \beta X$.*

1.17. Theorem. *([10]) $\mathcal{L}$-$\dim X \leqslant \dim X$ and $\mathcal{L}$-$\dim X = \dim X$ if and only if $\mathcal{L}$ contains a disconnected ANR-compactum L.*

1.18. Theorem. *([10]) If a hereditarily normal space X is the union of its subsets X_1 and X_2 such that $\mathcal{L}$-$\dim X_1 \leqslant m$ and $\mathcal{L}$-$\dim X_2 \leqslant n$, then*

$$\mathcal{L}\text{-}\dim X \leqslant m + n + 1.$$

1.19. Theorem. *([10]) If X is a metrizable space of finite dimension, then*

$$L\text{-}\dim (X \times I) \leqslant L\text{-}\dim X + 1$$

for an arbitrary ANR-compactum L.

2. Dimensional scales and their realization

2.1. Definition. Let L be an *ANR*-compactum. The set of all non-negative integers n such that there is a space X with $L\text{-}\dim X = n$ is called an *L-dimensional scale* and is denoted by *L-d-sc*.

2.2. Proposition. *L-d-sc $= \{0\}$ if and only if L is contractible.*

Proof. If L-*d*-*sc* $= \{0\}$, then $L \in AE(X)$ for every space X, in particular for $X = I^\infty$. There is an embedding $L \subset I^\infty$. The condition $L \in AE(I^\infty)$ implies that there is a retraction $I^\infty \to L$. Thus L is contractible.

On the other hand, if L is contractible, then L is an AR-compactum. Hence $L \in AE(X)$ for every space X, i.e. $L\text{-}\dim X \leqslant 0$. □

2.3. Proposition. *L-d-sc $= \{0, 1\}$ if and only if L is not contractible, but $L * L$ is contractible.*

Proof. $\Rightarrow$. If $1 \in L$-*d*-*sc*, then L is not contractible by Proposition 2.2. Further, the equality L-*d*-*sc* $= \{0, 1\}$ implies that $L\text{-}\dim X \leqslant 1$ for every X. Consequently, $L * L \in AE(X)$ for every X. Hence $L * L$ is contractible.

$\Leftarrow$. If $L * L$ is contractible, then $L * L \in AE(X)$ for every X which implies $L\text{-}\dim X \leqslant 1$. Hence L-*d*-*sc* $\subset \{0, 1\}$. The condition

$$L \quad \text{is not contractible}$$

and Proposition 2.2 imply that L-*d*-*sc* $\neq \{0\}$. □

2.4. Remark. There exists a two-dimensional non-contractible polyhedron L such that $L * L$ is contractible (look at [11]).

2.5. Theorem. *If $L * L$ is not contractible, then L-d-sc $= \omega$. Moreover, for every $n \geqslant 0$ there is $m \geqslant 0$ such that $L\text{-}\dim I^m = n$.*

To prove this theorem we need several auxiliary assertions.

2.5.1. Statement. *If $n \in L$-d-sc, then there exists a compactum C with $L\text{-}\dim C = n$.*

Proof. Since $n \in L$-*d*-*sc*, there exists a space X with $L\text{-}\dim X = n$. Then $L\text{-}\dim \beta X = n$ by Theorem 1.16. According to Theorem 1.15 βX is the limit space of an inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ such that all X_α are compacta with $L\text{-}\dim X_\alpha \leqslant n$. Then $L\text{-}\dim X_\alpha = n$ for some α. Otherwise $L\text{-}\dim X \leqslant n - 1$ in view of Theorem 1.14. □

2.5.2. Statement. *If X is a separable metrizable space with $L\text{-}\dim X = n$, then for every $m \in [0; n]$ there is a subspace $Y \subset X$ such that $L\text{-}\dim Y = m$.*

Proof. By Theorem 1.9 X can be represented as the union

$$X = X_1 \cup \cdots \cup X_{n+1} \tag{2.1}$$

so that $L\text{-}\dim X_i \leqslant 0$ for all i. Put $Y = X_1 \cup \cdots \cup X_{m+1}$. Then $L\text{-}\dim Y = m$. Indeed, $L\text{-}\dim Y \leqslant m$ according to Theorem 1.18. Suppose $L\text{-}\dim Y \leqslant m - 1$. Then (2.1) and Theorem 1.18 imply that $L\text{-}\dim X \leqslant n - 1$. □

Statements 2.5.1 and 2.5.2 yield

2.5.3. Statement. *If $n \in L$-d-sc, then $m \in L$-d-sc for every $m \in [0, n]$.*

2.5.4. Statement. *$n \in L$-d-sc if and only in $\overset{n}{*} L$ is not contractible.*

Proof. $\Rightarrow$. If $\overset{n}{*} L$ is contractible, then $\overset{n}{*} L \in AE(X)$ for every X. Hence $L\text{-}\dim X \leqslant n - 1$ for every X. Consequently, $n \notin L$-*d*-*sc*.

$\Leftarrow$. Take a polyhedron M such that $M \simeq L$. Then $\overset{n}{*} M$ is not contractible. Let $n * ^8 M \subset I^m$. Since $n * M$ is not contractible, there is no retraction $r : I^m \to n * M$. Hence $n * M \notin AE(I^m)$. Thus $M\text{-}\dim I^m \geqslant n$. On the other hand $M\text{-}\dim I^m \leqslant m$ in view of Theorem 1.17. Consequently, according to Statement 2.5.2 there is a subspace $Y \subset I^m$ such that $M\text{-}\dim Y = n$. It follows from Proposition 1.13 that $L\text{-}\dim Y = n$. □

2.5.5. Statement. *It $L * L$ is not contractible, then $\overset{n}{*} L$ is not contractible for every $n \geqslant 2$.*

Proof. It is known (look at [13]) that

$$\tilde{H}_p(L_1 * L_2) = \sum_{r+s=p-1} \tilde{H}_r(L_1) \otimes \tilde{H}_s(L_2) \oplus \sum_{r+s=p-2} \operatorname{Tor}\bigl(\tilde{H}_r(L_1), \tilde{H}_s(L_2)\bigr) \tag{2.2}$$

for *ANR*-compacta L_i.

We claim that

$$\tilde{H}_*(L * L) \neq 0. \tag{2.3}$$

In fact, if L is disconnected, then (2.2) implies that $\tilde{H}_1(L * L) = \mathbb{Z} \oplus G$. If L is connected, then $L * L$ is one-connected. Suppose $\tilde{H}_*(L * L) = 0$. Then $\pi_*(L * L) = 0$ by Hurewicz theorem and, consequently, $L * L$ is contractible according to Whitehead theorem. So condition (2.3) is verified. It remains to notice that (2.2) and (2.3) imply

$$\tilde{H}_*\bigl(\overset{n}{*} L\bigr) \neq 0$$

for every $n \geqslant 2$. Hence $\overset{n}{*} L$ is not contractible. □

Proof of Theorem 2.5. $\overset{n}{*} L$ is not contractible for every $n \geqslant 2$ by Statement 2.5.5. Thus $n \in L\text{-}d\text{-}sc$ in view of Statement 2.5.4.

To prove the second part of the assertion, we can assume that L is a polyhedron. There exists k such that $\overset{n}{*} L \subset I^k$. Since $\overset{n}{*} L$ is not contractible, $\overset{n}{*} L \notin AE(I^k)$. Consequently, Theorem 1.17 yields

$$L\text{-}\dim I^n \leqslant n \leqslant L\text{-}\dim I^k \leqslant k. \tag{2.4}$$

Inequalities (2.4) and Theorem 1.19 imply that $L\text{-}\dim I^m = n$ for some $m \in [n; k]$. □

3. Fully closed mappings

Let $f : X \to Y$ be a mapping and $A \subset X$. Recall that the set

$$f^{\#}A = \bigl\{y \in Y\colon\ f^{-1}(y) \subset A\bigr\} = Y \setminus f(X \setminus A)$$

is said to be the *small image* of A. If α is a family of subsets of X then we put $f^{\#}\alpha = \{f^{\#}A\colon\ A \in \alpha\}$.

3.1. Definition. ([3]) A continuous mapping $f : X \to Y$ is called *fully closed* if for every point $y \in Y$ and for every finite family u of open sets in X with property $f^{-1}(y) \subset \bigcup u$, the set $\{y\} \cup (\bigcup f^{\#}u)$ is a neighborhood of y.

1Obviously every fully closed mapping is closed.

3.2. Proposition. *If $f : X \to Y$ is a fully closed mapping and u is a finite open cover of X, then the set $Y \setminus \bigcup f^{\#}u$ is discrete.*

3.3. Proposition. *If $f : X \to Y$ is a fully closed mapping and $Z \subset Y$, then the mapping $f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z$ is fully closed.*

3.4. Proposition. *If $f : X \to Y$ and $g : Y \to Z$ are mappings whose composition $g \circ f$ is fully closed, then the mapping g is also fully closed.*

3.5. Definition. A quotient mapping $f : X \to Y$ is said to be *elementary* if the preimages of all (or of all but one) points of Y are one-point.

Thus, the space Y is obtained from X by contracting some closed set $F \subset X$ to a point.

3.6. Theorem. *([7, Theorem II.3.8]) Any compact fully closed mapping $f : X \to Y$ decomposes into a continuous well ordered inverse system $\{X_\alpha, f_\beta^\alpha, A\}$ in which all neighboring bonding mappings $f_\alpha^{\alpha+1}$ are elementary and their non-one-point fibers are homeomorphic to the fibers of the mapping f.*

3.7. Proposition. *Let $f : X \to Y$ be an elementary mapping and let $f^{-1}(y_0)$ be the only non-one-point fiber. Then*

$$\mathcal{L}\text{-}\dim X \leqslant \max\bigl\{\mathcal{L}\text{-}\dim Y, \mathcal{L}\text{-}\dim f^{-1}(y_0)\bigr\}. \tag{3.1}$$

Proof. The pair $(X, f^{-1}(y_0))$ satisfies conditions of Proposition 1.11. □

For a mapping $f : X \to Y$ the number $\mathcal{L}\text{-}\dim f$ is defined as follows

$$\mathcal{L}\text{-}\dim f = \sup\bigl\{\mathcal{L}\text{-}\dim f^{-1}(y)\colon\ y \in Y\bigr\}.$$

3.8. Theorem. *If $f : X \to Y$ is a fully closed mapping between compact spaces, then*

$$\mathcal{L}\text{-}\dim X \leqslant \max\{\mathcal{L}\text{-}\dim Y, \mathcal{L}\text{-}\dim f\}. \tag{3.2}$$

Proof. There exists a continuous well ordered inverse system $S = \{X_\alpha, f^\alpha_\beta, A\}$, $A = [0, \delta)$, satisfying conditions of Theorem 3.6. In particular, $X_0 = Y$, $X = \lim S$ and $f = f_0$, where $f_0 : \lim S \to X_0$ is the limit projection. Let $\max\{\mathcal{L}\text{-}\dim Y, \mathcal{L}\text{-}\dim f\} = n$. In accordance with Theorem 1.14, to prove (4.2), it suffices to show that

$$\mathcal{L}\text{-}\dim X_\alpha \leqslant n. \tag{3.3}$$

We do it by transfinite recursion on α. To pass from α to $\alpha + 1$ we apply Proposition 3.7. To pass to limit α we use Theorem 1.14. □

In applications, fully closed mappings appear as resolutions.

3.9. Definition. ([7]) Suppose given a space X, spaces Y_x, and continuous mappings $h_x : X \setminus \{x\} \to Y_x$ for each point $x \in X$. A *resolution* of (the set) X (at each point x to the space Y_x by means of the mappings h_x) is the set

$$R(X) \equiv R(X, Y_x, h_x) = \bigcup \bigl\{\{x\} \times Y_x \colon x \in X\bigr\}.$$

The mapping $\pi = \pi_X : R(X) \to X$, taking (x, y) to x, is called the *resolution mapping* or, simply, *resolution*.

Define a topology on $R(X)$. Given a triple (U, x, V) where U is an open subset of X, $x \in U$, and V is an open subset of Y_x, put

$$U \otimes_x V = \{x\} \times V \cup \pi^{-1}\bigl(U \cap h_x^{-1}(V)\bigr).$$

The family of sets of the form $U \otimes_x V$ is the base for some topology on $R(X)$ called the *resolution topology*.

3.10. Theorem. *([5]) If X and all Y_x are compact Hausdorff spaces, then $R(X)$ is also a compact Hausdorff space, π is fully closed, and each fiber $\pi^{-1}(x)$ is homeomorphic to Y_x. Moreover, $R(X)$ is first countable if and only X and all Y_x are first countable.*

3.11. Definition. A closed mapping $f : X \to Y$ is called *atomic* if $F = f^{-1} f(F)$ for every closed $F \subset X$ such that $f(F)$ is a continuum (connected closed non-singleton).

A number of applications of resolutions are based on the following statement.

3.12. Lemma. *([6,8]) Let X be a first countable connected compact Hausdorff space and let Y_x, $x \in X$, be an AR-compacta. Then we can choose mappings $h_x : X \setminus \{x\} \to Y_x$ so that*

(1) *the resolution $\pi_X : R(X) \to X$ is an atomic mapping.*

If X is perfectly normal and hereditarily separable then, under the continuum hypothesis, the mappings h_x can be chosen so that

(2) *in addition to* (1)*, the space $R(X)$ is perfectly normal and hereditarily separable.*

4. Vietoris–Begle theorem

Theorems 1.14 and 1.15 can be generalized as follows.

4.1. Theorem. *Let K be a CW complex. If X is the limit space of an inverse system $\{X_\alpha, \pi^\alpha_\beta, A\}$ of compact Hausdorff spaces X_α such that $K \in AE(X_\alpha)$ for every $\alpha \in A$, then $K \in AE(X)$.*

4.2. Theorem. *Let K be a countable CW complex and let X be a compact Hausdorff space such that $K \in AE(X)$. Then X is the limit space of a σ-spectrum $\{X_\alpha, \pi^\alpha_\beta, A\}$ such that $K \in AE(X_\alpha)$ for every $\alpha \in A$.*

The next theorem is a non-metrizable version of Dranishnikov's theorem from [2].

4.3. Theorem. *Let L be a simply connected countable CW complex and let X be a finite-dimensional compact Hausdorff space. Then the following conditions are equivalent*

(1) *$L \in AE(X)$; and*
(2) *$c\text{-}\dim_{H_i(L)} X \leqslant i$ for every integer $i \geqslant 0$.*

Proof. (1) ⇒ (2). By Theorem 4.2 X is the limit space of a σ-spectrum $\{X_\alpha, \pi^\alpha_\beta, A\}$ such that $L \in AE(X_\alpha)$. Then

$$c\text{-}\dim_{H_i(L)} X_\alpha \leqslant i \quad \text{for every integer } i \geqslant 0.$$

Applying Theorem 4.1 we get condition (2).

(2) ⇒ (1). Fix $i \geqslant 0$ Recall that $K(G,n)$ is a Filenberg–MacLane complex. Condition (2_i) implies that $K_i \equiv K(H_i(L), i) \in AE(X)$ (look at [12]).

Since L is countable, K_i is also countable. By Theorem 4.2 X is the limit of a σ-spectrum $S_i = \{X^i_\alpha, {}^i\pi^\alpha_\beta, A_i\}$ such that $K_i \in AE(X^i_\alpha)$ for every $\alpha \in A_i$. By Shchepin's spectral theorem [14], for every i, the indexing set A_1 contains an ω-complete cofinal subset A^i_1 such that A^i_1 is isomorphic to some ω-complete cofinal in A_i subset A^1_i and inverse systems

$$S^i_1 = \left\{X^1_\alpha, {}^1\pi^\alpha_\beta, A^i_1\right\} \quad \text{and} \quad S^1_i = \left\{X^i_\alpha, {}^i\pi^\alpha_\beta, A^1_i\right\}$$

are naturally homeomorphic. Put $A = \bigcap\{A^i_1\colon i \in \omega\}$ and $S = \{X^1_\alpha, {}^1\pi^\alpha_\beta, A\}$. Then condition $K_i \in AE(X^1_\alpha)$ is fulfilled for every $\alpha \in A$ and $i \geqslant 0$. Consequently,

$$c\text{-}\dim_{H_i(L)} X^1_\alpha \leqslant i \quad \text{for every } i \geqslant 0.$$

Hence $L \in AE(X^1_\alpha)$ for all $\alpha \in A$. Applying Theorem 4.1 we complete the proof. □

We need the following version of Vietoris–Begle theorem.

4.4. Theorem. *([15]) Let $f : X \to Y$ be an acyclic mapping of a compact Hausdorff space X anto a space Y. Then, for an arbitrary Abelian group G, the homomorphism*

$$f^* : H^q(Y, G) \to H^q(X, G)$$

is an isomorphism for any q.

As a corollary we get

4.5. Theorem. *Let $f : X \to Y$ be an acyclic mapping of a compact Hausdorff space X onto a space Y. Then*

$$c\text{-}\dim_G Y \leqslant c\text{-}\dim_G X.$$

Theorems 4.3 and 4.5 yield

4.6. Theorem. *Let L be a simply connected countable CW complex and let $f : X \to Y$ be an acyclic mapping from a finite-dimensional compact Hausdorff space X onto a finite-dimensional space Y. If $L \in AE(X)$, then $L \in AE(Y)$.*

Questtion 1. Does Theorem 4.6 hold for any countable CW complex L?

4.7. Theorem. *Let L be an ANR-compactum and let $f : X \to Y$ be an acyclic mapping from a finite-dimensional compact Hausdorff space X onto a finite-dimensional space Y. If L-$\dim Y \neq 1$, then L-$\dim X \geqslant L$-$\dim Y$. If L-$\dim Y = 1$, then L-$\dim X \geqslant 1$ for a simply connected L.*

Proof. If L-$\dim Y = 0$, then the assertion is obvious. Assume that L-$\dim Y = 1$, but L-$\dim X = 0$. Then $L \in AE(X)$ and, according to Theorem 4.6, $L \in AE(Y)$. Consequently, L-$\dim Y = 0$ and we arrive at a contradiction. At last, assume that L-$\dim Y = n \geqslant 2$, but L-$\dim X \leqslant n-1$. Then $\overset{n}{*} L \in AE(X)$. If L is connected, then $\overset{n}{*} L$ is one-connected. Hence $\overset{L}{*} \in AE(Y)$ by Theorem 4.6 and we again get a contradiction with L-$\dim Y = n$. If L is disconnected, then L-$\dim = \dim$ in view of Theorem 1.17 and $\dim Y \leqslant \dim X$ by Theorem 4.5. □

Questtion 2. Does Theorem 4.7 hold in dimension 1 for an arbitrary *ANR*-compactum L?

5. Intermediate dimensions

Let X be a compact Hausdorff space such that L-$\dim X = n \geqslant 2$. We say that X is a *space with no intermediate L-dimensions* if

$$\text{either} \quad L\text{-}\dim F \leqslant 0 \quad \text{or} \quad L\text{-}\dim F = n \tag{5.1}$$

for every closed set $F \subset X$.

5.1. Theorem. *Let L be a simply connected ANR-compactum such that $L * L$ is not contractible and let $n \geqslant 2$. Then*

(1) *there exists a first countable compact Hausdorff space Z_n^L such that L-dim $Z_n^L = n$ and Z_n^L is a space with no intermediate L-dimensions.*
Under the continuum hypothesis
(2) *there exists a perfectly normal hereditarily separable compact Hausdorff space $Z_{n,0}^L$ with no intermediate L-dimensions.*

Proof. (1) By Theorem 2.5 there is m such that $L\text{-}\dim I^m = n$. We construct an inverse system $S = \{Z_i, \pi_i^{i+1}\}$ such that $Z_0 = I^m$ and $X_{i+1} = R(X_i)$ is the resolution from Lemma 3.12(1) with $Y_x = I^m$, $x \in X_i$, and $\pi_i^{i+1} = \pi_{X_i} : R(X_i) \to X_i$. Let $Z_n^L = \lim S$ and let $\pi_i : Z_n^L \to X_i$ be the limit mappings. Then $L\text{-}\dim Z_n^L \leqslant n$ in view of Theorems 3.8, 3.10 and 1.14. Since mappings π_k^{k+1} are acyclic and atomic, all limit mappings π_i are acyclic and atomic. Consequently, according to Theorem 4.7 from $L\text{-}\dim X_0 = n$ it follows that $L\text{-}\dim Z_n^L = n$. Let $F \subset Z_n^L$ be a closed subset with $L\text{-}\dim F \geqslant 1$. Then F contains a continuum F_1. There is $i \in \omega$ such that $\pi_i(F_1)$ is a continuum. Since π_i is an atomic mapping, $F_1 = \pi_i^{-1}\pi_i(F_1)$. Take $x \in \pi_i(F_1)$ and put $F_0 = \pi_i^{-1}(x)$. Then $F_0 \subset F_1 \subset F$ and $L\text{-}\dim F_0 = n$. The proof of the last equality is similar to this of $L\text{-}\dim Z_n^L = n$.

(2) Instead of Lemma 3.12(1) we apply Lemma 3.12(2). □

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